Mathematical Finance
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Recall 3

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a complete filtration $\mathbb{F}$.

Good integrators and stochastic integrals

1. Let X be an $\mathbb{R}$-valued measurable process. For any $t \geq 0$, what is the stochastic integral up to time t with respect to X?
2. What is an $(\mathbb{F}, \mathbb{P})$-good integrator?
3. Let X be an $(\mathbb{F}, \mathbb{P})$-good integrator. Is the stochastic integral with respect to X uniquely defined? Which is the linearity property of $(\mathbb{F}, \mathbb{P})$-good integrators? Can you provide a proof for both questions?
4. Can you give three properties of an $(\mathbb{F}, \mathbb{P})$-good integrator? Can you prove them?

Stochastic integrals for unbounded integrands

1. Let X be an $(\mathbb{F}, \mathbb{P})$-good integrator. What is the class $\mathcal{L}^1(X, \mathbb{F}, \mathbb{P})$ of X-integrable processes?
2. Can you show that $\mathcal{L}^1(X, \mathbb{F}, \mathbb{P})$ is a vector space?
3. Let X be an $(\mathbb{F}, \mathbb{P})$-good integrator. For any $t \geq 0$, what is the stochastic integral up to time t with respect to X?
4. Let X be an $(\mathbb{F}, \mathbb{P})$-good integrator. Can you show that the stochastic integral on $\mathcal{L}^1(X, \mathbb{F}, \mathbb{P})$ is uniquely defined and linear with respect to its integrand?
5. Let X be an $(\mathbb{F}, \mathbb{P})$-good integrator, $\xi \in \mathcal{L}^1(X, \mathbb{F}, \mathbb{P})$, and set for any $t \geq 0$, $Y_t := \int_0^t \xi_s \mathrm{d}X_s$. Can you show that Y is an $\mathbb{F}$-adapted process, that the stochastic integral with respect to Y is well-defined over $\mathcal{P}_b(\mathbb{F}, \mathbb{P})$, and satisfies

$$\int_0^t Z_s \mathrm{d}Y_s = \int_0^t Z_s \xi_s \mathrm{d}X_s,\ t \geq 0,\ Z \in \mathcal{P}_b(\mathbb{F}, \mathbb{P})\,?$$

Relatively straightforward properties of stochastic integrals

1. When do the stochastic and the classical Lebesgue-Stieljes integrals coincide? Can you prove it?
2. Can you give two properties concerning stochastic integrals that are stopped at stopping times?
3. How do good integrators and integrable processes behave under localisation?
4. Can you state the dominated convergence theorem? Can you prove it?
5. Let X be an $(\mathbb{F}, \mathbb{P})$-good integrator, let $\xi \in \mathcal{L}^1(X, \mathbb{F}, \mathbb{P})$, and define $Y = \int_0^\cdot \xi_s \mathrm{d}X_s$. What are the jumps ΔY_t, for $t \geq 0$? What about the continuity and the $\mathbb{F}$-predictability of Y? Can you provide a proof for the tree questions?

Less straightforward properties of stochastic integrals

1. Can you give a necessary and sufficient condition for a càdlàg, $\mathbb{F}$-adapted process X to be an $(\mathbb{F}, \mathbb{P})$-good integrator?

(Semi-)martingales are good integrators

1. Is a càdlàg, $(\mathbb{F}, \mathbb{P})$-local martingale an $(\mathbb{F}, \mathbb{P})$-good integrator? Can you prove it?
2. What is an $(\mathbb{F}, \mathbb{P})$-semi-martingale?

Preservation of the local martingale property

1. When does a stochastic integral with respect an $(\mathbb{F}, \mathbb{P})$-local martingale remain an $(\mathbb{F}, \mathbb{P})$-local martingale?
2. Let X be an $(\mathbb{F}, \mathbb{P})$-local martingale, let $\xi \in \mathcal{L}^1(X, \mathbb{F}, \mathbb{P})$, and define $Y = \int_0^\cdot \xi_s dX_s$. Can you give three statements equivalent to "*Y is an $(\mathbb{F}, \mathbb{P})$-local martingale*"?

Preservation of the martingale property

1. What are the Burkholder-Davis-Gundy inequalities?
2. When does a stochastic integral with respect an $(\mathbb{F}, \mathbb{P})$-martingale remains an $(\mathbb{F}, \mathbb{P})$ martingale? Can you prove it?

The general case: σ-martingales

1. What is an $(\mathbb{F}, \mathbb{P})$-σ-martingale?
2. Let X be an $(\mathbb{F}, \mathbb{P})$-semi-martingale. Can you give two conditions equivalent to the statement "*X is an $(\mathbb{F}, \mathbb{P})$-σ-martingale*"?
3. Which are the conditions under which an $(\mathbb{F}, \mathbb{P})$-σ-martingale is an $(\mathbb{F}, \mathbb{P})$-local martingale (Ansel-Stricker Lemma)?

Quadratic variations for good integrators

1. What is an $\mathbb{F}$-stochastic partition π of $\mathbb{R}_+$? What is the mesh of a partition?
2. Given two processes X and Y, and some $\mathbb{F}$-stochastic partition of $\mathbb{R}_+$ π, what is the quadratic variation of X along π? What is the quadratic co-variation of X and Y along π?
3. Let X and Y be $(\mathbb{F}, \mathbb{P})$-good integrators. What can you say about the limit in the Émery topology of the quadratic (co)-variations when the mesh of any stochastic partitions goes to 0?
4. Let X and Y be $(\mathbb{F}, \mathbb{P})$-good integrators. Can you write the integration by parts formula?
5. What is the Kunita-Watanabe inequality?